

The University of Chicago

THE EVOLUTION OF BINARY STARS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY
AND ASTROPHYSICS

By
WILLIAM MARKOWITZ

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ABSTRACT

I. The present theories on the evolution of binary stars are reviewed.

II. The effect of tidal forces on the period P of a binary is discussed for various stellar models. For the polytrope $n=3$, P can increase, at most, by a factor of about 1.4. The increase is practically zero for the models of Jeans and of Milne.

III. The densities of the components of eclipsing variables was found to be independent of the relative separation. The conclusion drawn is that the components are not contracting.

IV. A discussion of the distribution of eclipsing binaries with respect to the relative separation indicates that their absolute separations may be decreasing.

V. It is shown that a secular decrease of mass cannot convert the most massive spectroscopic binaries known into visual binaries of the average mass of the latter class. If the periods of spectroscopic binaries are being lengthened through a secular decrease of mass then there should be a statistical relation of the form $M = D_1/P^{1/2}$, where M is the total mass of the binary and D_1 is a constant. It is shown, however, that M is independent of P . The average value of M of a visual binary is not much smaller than the average value for a spectroscopic binary.

VI. It is doubtful if binaries whose periods are greater than about 55 days and less than several years have been formed from long-period binaries through the effects of close encounters with other stars.

VII. A resisting medium increases P , a , and e of a binary. It is shown that these elements will decrease even if it is radiating away mass faster than it is picking it up.

VIII. Whether or not binaries can pick up enough interstellar matter to compensate for the loss by radiation cannot be decided definitely at present. Certain diffuse nebulae are probably dense enough to form O- and B-type stars.

The conclusion drawn from the data considered in this paper is that *if there are no forces acting on a binary which tend to increase P , a , and e other than those due to tides, secular decrease of mass and close encounters, then, in general, P , a , and e are decreasing.*

I. INTRODUCTION

The purpose of this investigation is to determine, if possible, whether spectroscopic binaries have evolved from visual binaries or vice versa. The following four hypotheses have been suggested to account for the formation of binary stars: (1) condensations around separate nuclei in a nebula; (2) fission of stars; (3) action of a resisting medium; (4) growth of a planetary system. We shall assume that binaries have been formed in all of these ways, though there is no direct astronomical evidence that any given binary has been so formed.

The only forces known at present which can influence the period P , semi-major axis a , and eccentricity e of a binary are those due to (i) tides; (ii) perturbations due to other stellar systems (including

close encounters); (iii) friction in a resisting medium; (iv) secular decrease of mass. We shall assume that these forces are, and have been, acting on binaries.

The first hypothesis on the formation of binaries will not be considered in detail, at this point, and we shall proceed to a discussion of the others, the fission theory first.

If the source of a star's energy is due only to its gravitational potential, then in the course of time it will contract. The moment of momentum of the star must remain constant, and the angular velocity, unless it were initially zero, would increase. The figures of equilibrium of rotating bodies have been the subject of extensive researches by several mathematicians, notably Colin Maclaurin, Karl Gustav Jacobi, Henri Poincaré, Sir George Darwin, Karl Schwarzschild, Alexander Liapounoff, and Sir James Jeans. Most of the work has been done on homogeneous, incompressible fluid models.

Maclaurin showed that for small values of $Q = \omega^2 / 2\pi k^2 \rho$ (ω being the angular velocity, k^2 the gravitational constant, and ρ the density), the figure of stable equilibrium was an oblate spheroid of revolution whose oblateness vanished with Q . These stable spheroids, as was shown later, exist for all values of Q lying between 0 and 0.1871. Jacobi then found a series of stable ellipsoids of three unequal axes for $0.1420 \leq Q \leq 0.1871$. For $Q = 0.1871$ the Jacobi ellipsoids and the Maclaurin spheroids coincide. Poincaré found a series of unsymmetrical figures of equilibrium, the so-called "pear-shaped," which exist for values of Q smaller than those giving the stable ellipsoids. Darwin showed that the ellipsoids and the pear shapes coincided when $Q = 0.1420$.

Let us consider, then, the sequence of figures of stable equilibrium of a series of rotating homogeneous incompressible fluid bodies having the same mass and density, rotating as rigid bodies, but having increasing moment of momentum. For zero moment of momentum (zero angular velocity) the figure is a sphere. With increasing moment of momentum, and ω , we have the spheroids of revolution. With still higher values of the moment of momentum, but with decreasing ω , the stability passes to the ellipsoids of Jacobi. These become more and more cigar-shaped as the moment of momentum increases and ω decreases until finally Poincaré's figures are reached.

It must not be supposed, as a consequence, that a contracting body will pass through the series of figures just mentioned, since ρ will increase with ω . But F. R. Moulton¹ has shown that such a body will become oblate as it contracts.

Let us consider the stability of Saturn. We shall assume that it is homogeneous, incompressible, etc. Saturn has a density of 0.72 (water = 1) and a rotation period of 10.25 hours, and for it $Q = 0.0983$. If an external force could be applied in such a manner as to reduce its rotation period to 7.43 hours, Q would equal 0.1871, and its figure would be one of the ellipsoids of Jacobi. From this it might be inferred that Saturn is not far from instability. But the actual case is quite different. If no external forces were applied, and increased rotation was due solely to contraction, then, as Moulton found, the figure of Saturn would not reach that just mentioned (for $Q = 0.1871$) until Saturn had a density of 21 and a period of rotation of 1.40 hours. Furthermore, the figure would not become pear-shaped ($Q = 0.1420$) until its density equaled 93 and its period 0.77 hours.²

The stability of the pear shapes was not determined by Poincaré. It was conjectured that, if they were stable, a furrow would be formed in the star as it kept on contracting, and that it would finally divide into two. If they were unstable then a star could never become pear-shaped, and a cataclysm, whose nature it is impossible to predict, would occur when the point was reached at which the Jacobi ellipsoids pass over to the pear shapes. Darwin put Poincaré's work in a numerical form and concluded that the pear-shaped figures were stable. According to Darwin, then, double stars were formed by fission, the two components revolving as a rigid body about each other and nearly in contact. The effect of further shrinkage of the components would be, through the agency of tidal forces, to increase P , a , and e .

Liapounoff, however, thought he had proved that the pear-shaped figures were unstable. Darwin and he were unable to come to an

¹ *Carnegie Institution of Washington Publications*, No. 107, p. 137, 1909.

² Eliminating ω and ρ from the expression for Q gives $Q = cV^2r/m \sin^2 i$, where c is a known constant, V is the equatorial velocity, r the radius, m the mass of the star, and i the angle between the axis of rotation of the star and the line of sight. The errors involved in the determination of V , r , and m (i is indeterminate) are so large that it is impossible to determine the stability of single stars from the foregoing point of view.

agreement on the matter, and the question of stability was left open until Jeans, in 1915, cleared up the discrepancy and showed that the figures were unstable. Nevertheless, Jeans, from a study of rotating liquid cylinders, a two-dimensional analogue of the problem here discussed, believes that short-period spectroscopic binaries have been formed by fission. Tidal forces being unable to account for more than a certain limited increase in P , a , and e , his views are that (a) binaries with periods greater than 55 days are due to separate condensations in a nebula, close encounters with other stars having served to reduce P and a ; (b) binaries with periods less than 55 days have been formed by fission, tidal forces and a secular decrease of mass having increased P , a , and e ; and encounters from passing stars being too rare to influence the orbits of such short-period systems.

If it is true, however, that the energy of a star is derived from the annihilation of matter, then, since a star that is radiating mass loses moment of momentum, the tendency toward fission will be eliminated unless, as Jeans believes, stars are subject to sudden contraction.

The third hypothesis concerning the formation of binary stars is a modification of the capture theory, but the latter term is somewhat misleading, and will be dropped. As long as there is no physical contact, two stars in space, not acted upon by exterior forces, cannot capture each other, no matter how close they approach. The effect of a resisting medium on a binary or on two independent stars, i.e., the orbital eccentricity, $e \geq 1$, is to reduce P , a , and e .

For the formation of a double star by a resisting medium a very close approach is not required; an approach of two independent stars in a nebula to within 100 astronomical units might be sufficient to form a binary with a period of the order of 1000 years. From consideration of the fact that out of 15,000 known visual double stars only about 1000 show any traces at all of orbital motion, Hertzsprung¹ decided that the median period of the catalogued double stars is probably thousands of years. Thus, on the basis of the friction theory binary stars would be formed with periods of several thousand years, the continued effect of the resisting medium serving to reduce P , a , and e .

¹ *Bulletin of the Astronomical Institutes of the Netherlands*, 1, 149, 1922.

The rarity of close approaches, coupled with the supposedly short life of a star, has hitherto made it seem improbable that this theory could account for the formation of a large number of binaries, especially those with short periods. But with the realization that the life of a star is much longer than previously supposed, and that it may even be indefinitely long, as will be pointed out in section viii, and in view of some of the factors mentioned in the preceding paragraph, it follows that this theory might be able to account satisfactorily for the large number of binaries known.

W. D. MacMillan has suggested that ultimately the solar system will become a binary. If the sun is losing mass by radiation, Jupiter, while picking up interstellar matter, is radiating but slightly. He believes that as the masses of Jupiter and the sun approach equality the other planets will probably fall into the sun or Jupiter, or suffer disruption from increased tidal strains. There is a possibility, of course, that a triple or multiple system may be formed. MacMillan has made a detailed study of the mathematical aspects of this hypothesis and of related cosmological questions.¹

The problem of two bodies with diminishing mass has been discussed by Jeans,² MacMillan,³ and E. W. Brown.⁴ They find, on the assumption that the law of motion is $\text{Force} = \text{Mass} \times \text{Acceleration}$, that the effect of a secular decrease of mass is to increase P and a , but to leave e invariant; P varying inversely as M^2 , M being the sum of the masses, and a varying inversely as M . It will be shown in section vii that P , a , and e may decrease even if a binary is radiating mass faster than it is picking it up in its journey through space.

According to the fission theory a binary in which the components are nearly in contact is in the earliest stages of its existence, while on the basis of the friction theory it is in the last stages. Offhand, one might think that it would be easy to decide between such contradictory theories, but that is not the case. Practically everything

¹ *Astrophysical Journal*, 48, 35, 1918; *American Mathematical Monthly*, 26, 326, 1919; *Science*, 52, 67, 1920; *Scientia*, 33, 3 and 103, 1923; *Science*, 62, 63, 96 and 121, 1925; *Monthly Notices of the Royal Astronomical Society*, 85, 904, 1924-1925.

² *Monthly Notices of the Royal Astronomical Society*, 85, 2, 912, 1924-1925.

³ *Ibid.*, p. 904.

⁴ *Proceedings of the National Academy of Sciences*, 11, 274, 1925; 12, 1, 1926.

that has been brought up as a confirmation of one of these theories could be shown to be equally consistent with the other. It is the purpose of this paper to investigate certain stellar statistics whose relations to theories of the evolution of binaries may be free from the objection just stated.

II. THE EFFECTS OF TIDAL FORCES

We have already mentioned Darwin's supposition that a binary, after having been formed by fission, would rotate as a rigid body; i.e., the orbital period and the rotation periods would all equal each other. He showed that further shrinkage of the components would, because of their increased angular velocity, set up a couple which would tend to increase P , a , and e .

Moulton,¹ however, by a consideration of the fact that the moment of momentum of the binary and its total energy (including the amount lost by tidal friction) must each remain constant, was able to show that tidal evolution, in so far as increasing P and a are concerned, is subject to very definite limits. He showed that in the case of binary stars where the masses of the components are comparable P could increase only by a small amount after fission.

In the case of binaries in which the components are comparable in mass, a consideration of the fact that the moment of momentum must remain constant is in itself sufficient to show how little tidal forces can increase P and a . The process of tidal evolution in this case is essentially a transference of the rotational moment of momentum of the components into the orbital moment of momentum. According to Darwin's theory the rotation periods of the components must always be equal to or less than the orbital period. In our discussion here we shall assume that the periods are always equal—the case most favorable to the fission theory. We shall also assume that the binary is not radiating any mass and that it is not acted upon by any forces which would tend to alter its moment of momentum.

Let A and B denote the two components, and let m_1 = the mass, a_1 = the distance from the center of A to the center of gravity of $A + B$, and r_1 = the radius of A . Similar symbols with the subscript 2 will refer to B . Further, let $M = m_1 + m_2$, $a = a_1 + a_2$, $\omega = 2\pi/P$ = the

¹ *Op. cit.*, pp. 106, 153.

angular velocity of the system ($A+B$) about the orbital axis of revolution, and I = the moment of inertia of the system about the same axis. The orbital eccentricity is small for close binaries, and is neglected. The value of the moment of momentum, which must remain constant, is

$$\omega I = \omega(m_1 a_1^2 + m_2 a_2^2 + a m_1 r_1^2 + a m_2 r_2^2) . \quad (1)$$

The coefficient a depends upon the distribution of density within the star. If the star is homogeneous $a = 0.4$. The value of a was computed for the polytropic gas sphere $n = 3$ from R. Emden's solution,¹ and found to equal 0.075. In this model, which Sir A. S. Eddington considers actually to be followed by the stars, the central density is 54.36 times the average density. For the liquid stars of Jeans and the models of E. A. Milne, which are highly condensed at the center, a is practically zero.

A study of the orbital elements indicates that the radii of the components are usually near equality. We shall assume, then, that r_1 equals r_2 , and drop the subscripts on these letters. Expressing m_1, m_2 in terms of M and the mass ratio c ; a_1, a_2 in terms of a and c , and eliminating ω from (1) by means of Kepler's equation,

$$\omega^2 a^3 = M k^2 ,$$

then, since the moment of momentum is constant, there results the equation

$$a = \frac{a_0(1+\beta)^2}{(1+\beta q^2)^2} , \quad (2)$$

where a_0 is the value of a when the components are in contact, and

$$\beta = \frac{a(1+c)^2}{4c} , \quad q = \frac{r}{a} . \quad (3)$$

The value of q when the components are in contact is evidently 0.5. The maximum value of a in equation (2) is obtained when q equals zero, i.e., when the components have shrunk to zero dimensions and have transferred all their rotational moment of momentum into

¹ *Gaskugeln*, Table 7, p. 80, 1907.

orbital. The value of the coefficient of a_0 in (2) has been computed for this case ($q=0$), for various values of α and c , and is entered in Table I.

The mass ratio, c , is rarely as small as 0.2, the average being about 0.75, and since the stars are probably as highly condensed toward

TABLE I

$c =$	1.00	.50	.20
$\alpha = .40$	1.96	2.10	2.96
.075.....	1.15	1.17	1.28
.00.....	1.00	1.00	1.00

the center as in Eddington's model or even more so, $\alpha \leq 0.075$. Hence, tidal forces can increase a , at most by a factor of about 1.3, and P , by Kepler's law, by a factor of about 1.4.

In the fission theory eclipsing variables of the β Lyrae type, whose light-curves vary continuously, are supposed to be binaries only

TABLE II

Class	$\log 10P$	P	β Lyrae	Algol	Sp. Binaries
I.....	.00	.100 ^d			
II.....	.50	.316	8	1	
III.....	1.00	1.00	41	27	8
IV.....	1.50	3.16	16	153	62
V.....	2.00	10.0	9	130	86
VI.....	2.50	31.6	3	24	51
VII.....	3.00	100	3	6	28
VIII.....	3.50	316	1	4	27
IX.....	5.52	33,000		2	41
Totals.....			81	346	303

recently formed. These are supposed to evolve into Algol-type variables and finally into spectroscopic binaries with periods up to about 55 days. In Table II, 81 β Lyrae variables, 346 Algol, and 303 spectroscopic systems have been divided into 9 groups according to period. Columns 2 and 3 contain the upper limit, respectively, of $\log 10P$ and P in each group. Columns 4, 5, and 6 give the number of systems of each class. The data for the eclipsing systems are taken

from R. Prager's *Katalog und Ephemeriden veränderlicher Sterne für 1931*,¹ and that for the spectroscopic binaries from A. Beer's *Zur Charakterisierung der spektroskopischen Doppelsterne*.²

More than 60 per cent of the β Lyrae stars have periods shorter than 1 day, while most of the Algol variables have periods between 2 and 10 days. On the other hand, there are almost as many spectroscopic binaries with periods greater than 10 days as there are with periods less than 10 days. Since tidal forces can increase P , at most, by three- or four-tenths of its value after fission, it follows that these forces cannot change the eclipsing variables into the ordinary spectroscopic binaries. In fact, from Table II it follows that tidal forces could not even change the β Lyrae variables into Algol variables. Since tidal forces decrease with the separation there should be no tendency for those members of a group which have the longer periods to evolve into the next group faster than those with the shorter periods.

III. THE DENSITIES OF ECLIPSING BINARIES

On the basis of the fission theory it is true that statistically the densities of the components A and B of a binary in which A and B are nearly in contact should be less than one in which the components are somewhat separated. Let the relative separation, σ , be defined as follows:

$$\sigma = \frac{\text{Distance between surfaces of } A \text{ and } B}{\text{Distance between centers of } A \text{ and } B}.$$

If A and B are in contact, $\sigma = 0$; while if A and B have shrunk to infinitesimal dimensions, $\sigma = 1$. There are two factors which can cause σ to increase: (1) an increase in the absolute separation (center to center); (2) contraction of A and B .

Let ρ_1 and ρ_2 be the densities of A and B , and $\rho_{1,0}$ and $\rho_{2,0}$ their values when $\sigma = 0$. Suppose that A and B contract but that the

¹ "Kleinere Veröffentlichungen der Universitäts Sternwarte zu Berlin-Babelsberg," No. 9, 1930.

² "Veröffentlichungen der Universitäts Sternwarte zu Berlin-Babelsberg," 5, Heft 6, 1927.

absolute separation does not change. Then, assuming that the ratio of the radii of A and B does not change,

$$\rho_i = \frac{\rho_{i,0}}{(1-\sigma)^3} \quad (i=1 \text{ or } 2). \quad (4)$$

Suppose, now, that tidal forces do increase the absolute separation and by the maximum amount possible. Then (2) must be satisfied. Eliminating a from (2) by means of the second of (3) and q by means of the relation,

$$\sigma = \frac{(a-2r)}{a} = 1-2q,$$

it is found that the expressions for the densities can be brought to the form

$$\rho_i = \frac{\rho_{i,0}[1+\beta(1-\sigma)^2]^6}{(1-\sigma)^3(1+\beta)^6}. \quad (5)$$

It is seen that when $\alpha=0$, $\beta=0$ and (5) reduces to (4).

If the mass ratio of an eclipsing binary is known, ρ_1 and ρ_2 can be computed from the orbital elements. The densities and relative separation of the components of 125 eclipsing binaries have been plotted in Figure 1. Of these 125, 90 are from H. Shapley's catalogue of eclipsing binaries.¹ The data for 14 of Shapley's list for which more recent orbits have been computed, and for 35 others not in his list, are given in Table III. Column 3 gives σ . The densities of the components ($\odot=1$) are given in columns 4 and 5. The sixth column gives the ratio of the densities, p ; it is always taken to be less than unity. The number in Shapley's catalogue is given in the first column.

The mass ratio was determined either from spectroscopic observations (when double-lined spectra are available) or from differences in brightness, or assumed to be unity. A small error in the mass ratio has little effect on the computed densities.

The average of the mean densities of 250 stars (125 binaries) was found to be $0.219 \times \odot$. The binaries were divided into classes, in Table IV, according to relative separation. Column 2 gives the upper

¹ *Contributions from the Princeton University Observatory*, No. 3, 1915.

limit of σ for the binaries in each class. Column 3 gives the mean density; 4, the number; 5, the average of the ratios of the densities; and 6, the number of binaries in each group for which $p < .10$. Four systems have $\sigma > .8$ and are not included in Table IV.



FIG. 1.—The densities of the components (sun=1) of 125 eclipsing binaries plotted against their relative separations. If the components were contracting, then the points should cluster around one of the above curves; which one, depends on the effectiveness of tidal forces. The values of ρ for large values of σ are not indicated on the diagram. For $\sigma=0.9$ the ordinates of the various curves are as follows: A, 210; B, 136; C, 97.5; D, 27.8.

In order to compare the observed densities with those to be computed by (4) and (5) it is necessary to determine ρ_0 . From Table IV it is seen that the average value of ρ for class I, which includes the systems in which A and B are in contact or nearly so, is 0.21. This

TABLE III
THE DENSITIES AND RELATIVE SEPARATION OF THE COMPONENTS
OF 49 ECLIPSING BINARIES

	Name	$\sigma = 1 - \frac{r_1 + r_2}{a}$	ρ_1	ρ_2	ϕ	Reference
	RV Ophiuchi	.64	.24	.06	.25	R. S. Dugan, <i>PO</i> , No. 4, 1916
	RZ Cassiopeia	.46	.45	.12	.27	<i>Ibid.</i>
5	U Cephei	.48	.21	.02	.10	<i>Ibid.</i> , No. 5, 1920
26	Y Camelopardi	.52	.06	.02	.33	<i>Ibid.</i> , No. 6, 1924
55	SZ Herculis	.33	.37	.17	.46	<i>Ibid.</i>
70	RS Vulpeculae	.53	.06	.008	.13	<i>Ibid.</i>
25	R Canis Majoris	.50	.51	.10	.37	<i>Ibid.</i>
	RV Aquarii	.57	.40	.06	.15	<i>Ibid.</i>
3	TV Cassiopeia	.41	.13	.05	.39	<i>Ibid.</i> , No. 7, 1924
	TW Cassiopeia	.67	.10	.18	.05	<i>Ibid.</i>
	TX Cassiopeia	.14	.02	.007	.33	<i>Ibid.</i>
	T Leonis Minoris	.55	.11	.02	.18	<i>Ibid.</i>
	SS Camelopardi	.40	.08	.005	.06	<i>Ibid.</i>
	X Trianguli	.37	.47	.14	.30	<i>Ibid.</i> , No. 8, 1928
4	RT Sculptoris	.20	1.80	.35	.10	<i>Ibid.</i>
	WW Aurigae	.66	.21	.20	.95	<i>Ibid.</i> , No. 10, 1930
	W Ursae Minoris	.27	.05	.05	1.0	<i>Ibid.</i>
13	RS Canum Venaticorum	.62	.45	.012	.03	B. W. Sitterly, <i>ibid.</i> , No. 11, 1930
	X Tauri	.48	.025	.024	.96	J. Stebbins, <i>AJ</i> , 51, 193, 1920
11	β Persei	.55	.15	.03	.20	<i>Ibid.</i> , 53, 105, 1921
	γ H Cassiopeia	.74	.28	.04	.15	<i>Ibid.</i> , 54, 83, 1921
40	δ Librae	.39	.08	.03	.38	J. Stebbins, <i>WO</i> , 15, 1928
	α Coronae Borealis	.934	3.8	.25	.07	<i>Ibid.</i>
	α Andromedae	.53	.15	.04	.27	C. M. Huffer, <i>ibid.</i>
	21 Cassiopeia	.75	.51	.12	.23	<i>Ibid.</i>
	H.D. 25833	.55	.25	.10	.40	C. M. Huffer, <i>unpublished</i>
	Boss 1607	.87	.50	.14	.36	<i>Ibid.</i>
	SX Hydrae	.60	.62	.02	.03	M. B. Shapley, <i>HB</i> , No. 707, 1924
	V Leporis	.18	.10	.08	.80	<i>Ibid.</i> , No. 812, 1927
	FP Carinae	.027	.016	.007	.44	<i>Ibid.</i> , No. 813, 1927
	ZZ Sagittarii	.51	.08	.02	.25	<i>Ibid.</i> , No. 847, 1927
	TT Hydrae	.60	.52	.005	.01	<i>Ibid.</i>
	UW Virginis	.50	.40	.05	.11	<i>Ibid.</i> , No. 848, 1927
	S Equulei	.71	.30	.07	.18	<i>Ibid.</i> , No. 874, 1930
	TW Andromedae	.50	.12	.02	.17	M. B. Shapley, <i>AJ</i> , 56, 430, 1922
	H.V. 3622	.73	.09	.02	.22	I. E. Woods and M. B. Shapley
1	SX Cassiopeia	.75	.09	.0005	.005	<i>HC</i> , No. 238, 1922
	SW Lacertae	.12	1.54	1.23	.80	B. P. Gerasimovic, <i>HB</i> , No. 852, 1927
	YY Geminorum	.65	2.2	1.0	.86	J. S. Schilt, <i>BAN</i> , 2, 175, 1924
	H.D. 1337	.02	.000	.003	.33	H. van Gent, <i>ibid.</i> , 6, 99, 1931
	TU Herculis	.55	.40	.03	.08	J. A. Pearce, <i>DO</i> , 3, 275, 1927
	SX Draconis	.49	.05	.007	.14	J. Q. Stewart, <i>AJ</i> , 42, 315, 1915
						W. van B. Roberts, <i>ibid.</i> , p. 312, 1915
87	RR Vulpeculae	.70	.30	.03	.10	M. Maggini, <i>ibid.</i> , 50, 141, 1910
	RT Lacertae	.45	.013	.013	1.0	M. Fowler, <i>ibid.</i> , 52, 257, 1920
	α Aquilae	.53	.15	.12	.83	C. C. Wylie, <i>ibid.</i> , 56, 232, 1922
	CG Cygni	.34	.75	.35	.47	Ch'ing-Sung Yu, <i>ibid.</i> , 58, 75, 1923
	S Antliae	.11	.38	.31	.82	A. H. Joy, <i>ibid.</i> , 64, 287, 1926
61	RX Herculis	.62	.31	.30	.97	R. F. Sanford, <i>ibid.</i> , 68, 51, 1928
19	TT Aurigae	.30	.12	.11	.91	Joy and Sitterly, <i>ibid.</i> , 73, 77, 1931

ABBREVIATIONS: *AJ* = *Astrophysical Journal*; *BAN* = *Bulletin of the Astronomical Institutes of the Netherlands*; *DO* = *Publications of the Dominion Astrophysical Observatory*; *HB* = *Harvard Bulletin*; *HC* = *Harvard Circular*; *PO* = *Contributions from the Princeton University Observatory*; *WO* = *Publications of the Washburn Observatory*.

value is near the average value for all the binaries and we shall put ρ_0 equal to it. Graphs of equations (4) and (5) have been drawn in Figure 1, using various values of α and c . The curves drawn are

- A. (4); or (5), $\alpha = 0.0$,
- B. (5), $c = 1$, $\alpha = .075$,
- C. (5), $c = .2$, $\alpha = .075$,
- D. (5), $c = 1$, $\alpha = 0.40$.

If no tidal action takes place or if $\alpha = 0$ (the latter case implies the former), the points in Figure 1 should cluster around curve A. If the maximum amount of tidal action does take place, then, since the

TABLE IV

Class	$\sigma = 1 - \frac{r_1 + r_2}{a}$	$\bar{p}$	n	$\bar{p}$	u
I.....	.1	.211	6	.61	1
II.....	.2	.422	7	.82	0
III.....	.3	.520	8	.51	1
IV.....	.4	.167	11	.42	2
V.....	.5	.137	21	.45	1
VI.....	.6	.112	29	.22	4
VII.....	.7	.254	22	.30	10
VIII.....	.8	.197	17	.28	9

mass ratio in binaries is about 0.75, the points should cluster around D, if the stars are homogeneous, and around B if their law of density is that given by Emden's polytrope, $n = 3$. Since it is obvious that none of these occurs, the conclusion drawn is that *the components of close binaries are not contracting*.

The results of this and the preceding sections show that tidal effects are negligible in the evolution of binary stars in which the masses of the components are comparable.

IV. FREQUENCY OF ECLIPSING BINARIES

Column 4 of Table IV shows the number of eclipsing variables in the various σ -classes. Contrary to expectation, most of these systems are not found in the classes having small values of σ (small separation between the surfaces). Curve A in Figure 2 is a smoothed

frequency-curve showing the number of systems to be found for an interval in σ of 0.1.

Let the two components of an eclipsing binary be equal in size and in luminosity, and let us assume that an eclipse will be visible to the

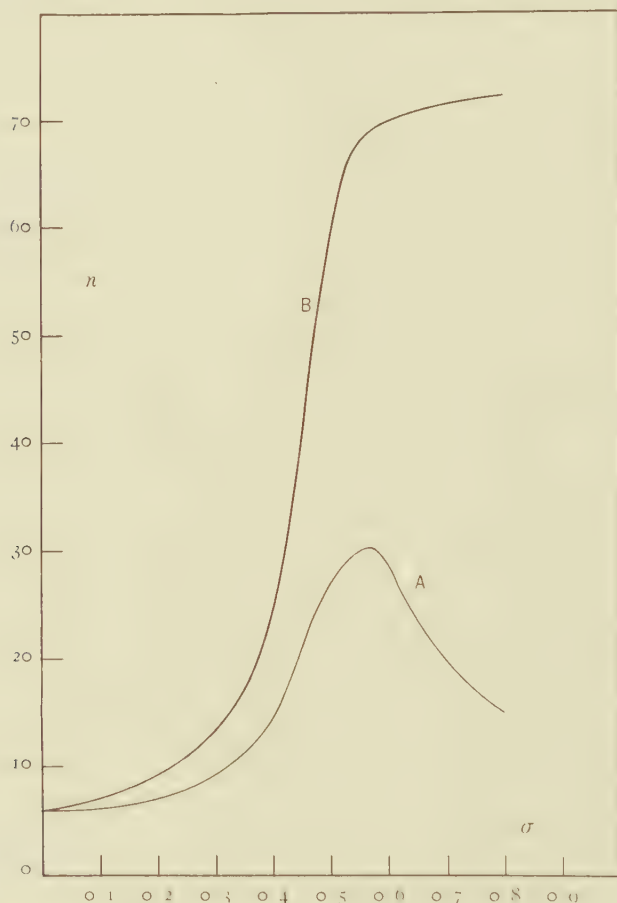


FIG. 2.—The distribution with respect to the relative separation of the eclipsing systems plotted in Figure 1 is given by curve *A*. The ordinates of *B* are proportional to the actual distribution.

earth if, at time of minimum, the portion of the radius of each star which is hidden $\geq l$, l being a constant. The orbital eccentricities of eclipsing binaries are small and will be neglected. Let i be the com-

plement of the angle between the line of sight and the normal to the plane of the orbit. Let j be the maximum value of i for which an eclipse will be visible.

From Figure 3 it is seen that

$$\sin j = \frac{r(2-l)}{a} = \frac{(1-\sigma)(2-l)}{2}.$$

The chance that $i \leq j$ is the same as the ratio of the area of the zone of a sphere lying between latitudes $+j$ and $-j$ is to the area of the sphere. Denoting this ratio by f , we have

$$f = 4\pi \int_0^j \frac{\cos \theta d\theta}{4\pi} = \frac{(1-\sigma)(2-l)}{2}.$$

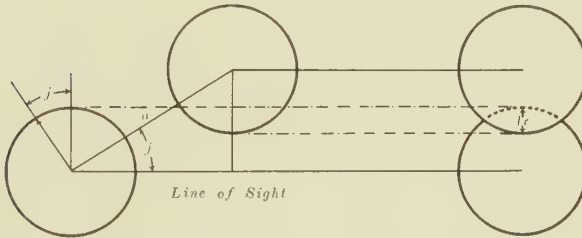


FIG. 3.—The configuration of the binary at the time of maximum eclipse

Hence, under the assumptions made, the chance that a given spectroscopic binary will be detected as an eclipsing system is proportional to $(1-\sigma)$. In order, then, to find the true distribution with respect to the relative separation, it is necessary to multiply all the ordinates of curve *A* in Figure 2 by $1/(1-\sigma)$. The resulting curve is labeled *B*. There is seen to be a sudden drop in the frequency as σ becomes less than .5.

Darwin, in a paper entitled "On the Figure and Stability of a Liquid Satellite,"¹ discussed the question of how close two equal liquid homogeneous incompressible masses could be brought to each other before tidal strains disrupted the bodies. From his results it is found that this happens when σ reaches a value of about 0.32. Stars actually found in nature increase in density toward the center,

¹ *Philosophical Transactions of the Royal Society, A*, 206, 161, 1906.

and as this factor leads to greater stability, two equal stars can probably approach each other more closely than when $\sigma=0.32$. On the other hand, if one star is much denser than the other, the latter may break up before σ gets as low as 0.32.

A glance at Figure 1 shows that for small values of σ the densities of the components of a given star are nearly equal (the two components have the same abscissa). Column 5 of Table IV gives the average value of p , the ratios of the densities, for each group, and 6 gives u , the number of binaries in each group for which $p < .10$.

There is an abrupt change in p and u as σ passes from .6 to .5. Hence, the conclusion that might be inferred from a study of Figure 2 and Table IV is that the components of close binaries are being drawn together, systems in which one component is much denser than the other being disrupted first.

V. SECULAR DECREASE OF MASS; VISUAL BINARIES

We have already stated that a secular decrease of mass will increase the orbital period, P of a binary; P varying inversely as the square of M , the sum of the masses. In Table V, P and M (present values) are tabulated for 22 short-period spectroscopic binaries whose inclinations are known. The data are taken from Table 45 of Beer's catalogue.¹ The masses that these systems would have after losing enough mass to become visual binaries with periods of 40 and 100 years, respectively, have been computed and entered in columns 5 and 6. The formulae used in their computation are

$$M(i) = M \left(\frac{P}{365.25i} \right)^{\frac{1}{2}} \quad (i = 40 \text{ or } 100).$$

R. G. Aitken² finds that the mean M of 14 visual binaries whose periods are of the order of 40 years is $1.76 \times \odot$. It is seen, then, that these spectroscopic binaries, which include some of the most massive known, could hardly hope to become long period visual binaries of the type now known, through a loss of mass.

SPECTROSCOPIC BINARIES

Tidal forces being incompetent to increase P and a , and close encounters from passing stars being admittedly a negligible factor in

¹ *Op. cit.*

² *The Binary Stars*, p. 209, 1918.

the evolution of close spectroscopic binaries, it remains for a secular decrease of mass to account for the increase in P and a after fission. If, indeed, P has been increased because of a loss of mass, then the following relation should be satisfied statistically, since, for any given binary, P varies inversely as M^2 :

$$\overline{M}\overline{P}^{\frac{1}{2}} = D_1, \quad (6)$$

D_1 being a constant. In applying this formula the range in P must be small for each separate group.

TABLE V

No. in Beer's Catalogue	Name	P	M	$M(40)$	$M(100)$
6.....	H.D. 1337	3.523 ^d	70.1 × ⊙	1.090 × ⊙	.689 × ⊙
22.....	V Puppis	1.454	35.6	.355	.231
23.....	Y Cygni	2.996	31.9	.451	.286
35.....	η Herculis	2.051	10.3	.122	.077
41.....	Z Vulpeculae	2.455	7.6	.099	.062
49.....	U Coronae Borealis	3.452	5.9	.090	.057
57.....	U Ophiuchi	1.677	10.0	.107	.067
61.....	β Lyrae	12.92	23.4	.695	.440
75.....	RS Vulpeculae	4.477	5.7	.100	.063
76.....	σ Aquilae	1.950	11.3	.158	.083
78.....	β Persei	2.867	5.7	.079	.051
89.....	RX Herculis	1.779	1.9	.021	.013
95.....	β Aurigae	3.960	4.7	.078	.049
99.....	TV Cassiopeiae	1.813	2.7	.031	.019
129.....	ξ_1 Ursae Majoris	20.54	7.5	.283	.179
134.....	TX Herculis	2.060	3.9	.046	.029
160.....	WW Aurigae	2.525	4.1	.054	.034
164.....	S Antliae	.648	1.2	.008	.005
180.....	Z Herculis	3.993	2.9	.048	.031
184.....	RS Canum Venaticorum	4.798	2.5	.046	.029
218.....	W Ursae Majoris	.334	1.2	.006	.004
279a.....	YY Geminorum	.814	1.2	.009	.006

For spectroscopic binaries we have the following well-known relation

$$KP^{\frac{1}{3}} = \frac{cC_0M^{\frac{1}{3}}\sin i}{(1+c)(1-e^2)^{\frac{1}{2}}}, \quad (7)$$

where K is the semi-amplitude of the velocity of the more massive component, C_0 is a constant, and c is the mass ratio. In a statistical study we may treat i , c , and e as constants. Hence, equation (7) becomes

$$\overline{K}\overline{P}^{\frac{1}{3}} = \overline{M}^{\frac{1}{3}}C_1, \quad (8)$$

where C_1 is a constant. If (6) is satisfied, then (8) must become

$$\bar{K}\bar{P}^{\frac{1}{2}} = D, \quad (9)$$

where D is a constant.

O. Struve¹ has grouped 116 spectroscopic binaries into 9 period groups whose mean periods ranged from 2.43 to 3199 days, and he found that

$$\bar{K}\bar{P}^{\frac{1}{2}} = C, \quad (10)$$

C being a constant independent of the period group and having a value of 116.7. This implies, from (8), that M is independent of the period classification.

TABLE VI
THE PERIOD AMPLITUDE FUNCTIONS

Class	Group	$\log 10P$	P	$\bar{P}$	$\bar{K}$	n	$C = \bar{K}\bar{P}^{\frac{1}{2}}$	$D = \bar{K}\bar{P}^{\frac{1}{2}}$
I.....	1	.30	.1995 ^d					
	2	.60	.3981	.347	143.7	3	101.0	85.7
	3	.90	.7943	.693	81.6	3	72.2	68.0
II.....	4	1.20	1.585	1.267	90.0	18	97.5	102.5
	5	1.50	3.102	2.302	84.4	45	112.3	120.7
	6	1.80	6.310	4.391	61.4	55	100.3	128.3
	7	2.10	12.59	9.430	52.2	48	110.3	160.2
	8	2.40	25.12	17.77	51.4	28	134.4	216.0
	9	2.70	50.12	34.71	36.2	20	118.0	213.0
	10	3.00	100.0	67.15	29.5	15	120.0	241.7
III.....	11	3.30	199.5	138.3	32.2	19	166.8	379
	12	3.60	398.1	260.9	18.7	9	119.2	302
	13	3.90	794.3	561.0	12.0	11	99.2	284
	14	4.20	1585	1027	15.1	10	152.5	484
	15	4.50	3102	2423	10.1	7	136.3	496
	16	4.80	6310	4111	14.4	5	231.0	924
	17	5.10	12,590	9350	7.25	4	152.8	700
	18	5.40	25,120	18,260	3.0	1	79.1	405
	19	5.70	50,120	30,840	2.0	2	62.7	350

In Beer's catalogue² 303 stars are available for discussion. They have been divided into 10 groups according to period as listed in columns 3 and 4 of Table VI. The interval in $\log 10P$ is taken to be 0.3. The values of P and $\log 10P$ are the upper limits for the systems

¹ *Astrophysical Journal*, 60, 167, 1924.

² *Op. cit.*

in each group. The 19 groups have been combined into 3 classes as indicated in column 1. In the other columns we have

2. the group number ,
5. $\bar{P}$,
6. $\bar{K}$,
7. n , the number in the group ,
8. $C = \overline{K\bar{P}^3}$,
9. $D = \overline{K\bar{P}^{\frac{1}{2}}}$.

Let C_i and D_i denote the values of C and D for each of the 19 groups. It is evident that there are not sufficient systems in the groups of classes I and III to give accurate individual values of C_i and D_i , but the results are tabulated for the sake of completeness. The weighted mean of the C_i , calling the number of stars in each group the weight, is 117.3. For the weighted means of the 3 classes we have

$$\begin{aligned} C \text{ (I)} &= 86.6 , \\ C \text{ (II)} &= 111.1 , \\ C \text{ (III)} &= 143.8 . \end{aligned}$$

the number of systems in each class being 6, 229, and 68, respectively.

It will be seen that in class II, where most of the spectroscopic binaries are found, the values of the C_i are quite uniform while the D_i increase with the period, and cannot be called constant. Hence, equation (9) is not satisfied but (10) is, and therefore, from (8), we can say that the average mass of a spectroscopic binary is independent of the period. Since (9) is not satisfied, (6) is not satisfied, and it follows that long-period spectroscopic binaries have not been formed from the short-period binaries through a secular decrease of mass.

The difference between $C \text{ (III)}$ and $C \text{ (II)}$ is probably due to the fact that K is small when $P > 1000$ days, and the tendency will be to observe those systems for which K is larger than the average.

If we put $\sin^3 i = 0.65$, $c = 0.75$, and neglect e , it is found, from (7), that the average total mass of the spectroscopic binaries listed in Table VI is $3 \times \odot$. We have already seen that the average mass of a

visual binary is $1.76 \times \odot$. In observing visual binaries the tendency will be to observe dwarfs that are close to the sun, while in the case of the spectroscopic binaries the tendency will be to observe distant giants. Since the giants are more massive than the dwarf stars, it follows that the difference between the masses of visual and spectroscopic binaries is small.

VI. CLOSE ENCOUNTERS

According to our first hypothesis in section i relating to the formation of binary stars, matter in a nebula condenses around two distinct nuclei and forms a visual binary with a long period. The period is certainly greater than several hundred years and probably runs into the thousands or tens of thousands. Binaries formed by the third hypothesis will also have very long periods.

TABLE VII

SPECTROSCOPIC BINARIES		VISUAL BINARIES	
P	e	P	e
$\leq 3^d$	.05	20^y	.4
12.....	.2	75.....	.5
50.....	.3	135.....	.55
1000.....	.35	275.....	.6

The effect of a close encounter of a star with such a long-period system is either to break it up, to increase the period, or to decrease the period. If the original eccentricity were zero, the effect of an encounter necessarily would be to increase it. Now, it is well known that, statistically, the eccentricity increases with the period. Roughly speaking, the correlation runs about as shown in Table VII.

As may be seen from column 7 of Table VI, most of the spectroscopic binaries have extremely short periods, over 50 per cent being less than 12.59 days and over 70 per cent being less than 50.12 days. Several close encounters would have to take place in order to reduce the period of a binary of several hundred years to as low as only a few days, and, when the frequency of close encounters is taken into consideration, it is obvious that most short-period spectroscopic binaries could not have been formed in the manner just stated. Furthermore, it is doubtful if a binary could undergo many close encounters without being disrupted.

Let us now consider the possibility of systems with P greater than 55 days and up to about 25 or 100 years having been formed from longer-period binaries through close encounters.

The eccentricities and periods of 362 double stars are listed in Table VIII. The data for the spectroscopic binaries are taken from Beer's catalogue,¹ and that for the visual pairs is from W. van den Bos's table of orbits.² Of the 303 spectroscopic systems in Beer's list,

TABLE VIII
THE DISTRIBUTION OF BINARY STARS WITH RESPECT
TO PERIOD AND ECCENTRICITY

GROUP LOG $10P$ P e	1	2	3	4	5	6	7	8	9	10	11	TOTALS
	.60	1.20	1.80	2.40	3.00	3.60	4.20	4.80	5.40	6.00	6.60	
	.4 ^d	1.59	6.31	25.12	100.0	400	4.35 ^y	17.2	68.6	273	1160	
.1.....	I	10	67	26	7	9	3					123, 0
.2.....		I	13	9	2	5	4	, I	, 7	2	I	34, 11
.3.....			I	3	15	3	I	2	4, I	, 3	3	29, 7
.4.....			I		7	5	I	3	2, 2	I, 10	6	20, 18
.5.....				3	3	2	5	, 2	, 7	14		13, 23
.6.....				7	3	3	I		, 7	13	3	14, 23
.7.....				3		I	I	I, I	I, I	4	I	7, 7
.8.....					3	I	I	, I	, 3	7	2	5, 13
.9.....						I			, 3	4	I	I, 8
1.0.....									, I	3	2	0, 6
Totals..	I	13	83	70	26	24	20	7, 8	2, 42	56	10	246, 116

e has not been determined for 49, and 8 others are also visual binaries, the value of e being determined by visual observations. These 8 are in van den Bos's table. One star in the list of 117 of van den Bos, Capella ($P=104.02d$, $e=.0086$, from interferometer measures of P. W. Merrill), has been treated as a spectroscopic binary. The values of $\log 10P$ and P and of e denote the upper values in the group. The entries in the table in italicized numerals refer to visual systems, the others to spectrographic.

The binaries to the right of the heavy line in Table VIII, inserted between groups 5 and 6, all have periods greater than 100 days, and, except possibly for a few rare cases, should have originated as long-

¹ *Op. cit.*

² *Bulletin of the Astronomical Institutes of the Netherlands*, 3, 149, 1926.

period systems. If the short periods of the binaries listed in groups 6, 7, and 8 are due to the effect of stellar encounters, then these systems must have had more encounters than those in groups 9, 10, and 11; and those in group 6 most of all.

Hence, since the average effect of encounters is to increase e , the average value of e as we go from the systems of group 11 to those of group 6 should increase. Actually, as we see from Table VII, the opposite is true. Referring to group 6 of Table VIII, we see that for more than one-half of the systems there $e \leq .2$, while for more than one-third $e \leq .1$. These latter stars are:

Boss No.	System	P	e
	H.D. 129132	101 ^d	.10
4934.....	ν Saggiarii	137.9	.09
967.....	μ Persei	284	.06
1246.....	Capella	104.0	.02
	H.D. 184398	108.6	.05
5192.....	22 Vulpeculae	251.0	.04
1082.....	3 Camelopardi	121	.02
329.....	γ Phoenicis	193.8	.01
1773.....	A Carinae	195.3	.0

That the perturbations of a passing star could affect a binary in such a way as to make its orbit circular is possible but highly improbable. The fact that such a large portion of the binaries with periods between 100 and 400 days have very low eccentricities leads us to the conclusion that these systems, in the main, have not been formed from long-period systems through the effects of stellar encounters.

VII. EFFECTS OF A RESISTING MEDIUM

A resisting medium will reduce P , a , and, as we shall later see, e of a binary. MacMillan¹ has found that for a binary picking up matter in space, P varies inversely as the fifth power of M , the sum of the masses, and a varies inversely as the cube of M . Thus, if a star radiates no mass, binaries will undergo a secular decrease in P , a , and e .

Suppose, now, that a binary is picking up mass at a rate of dM_1/dt

¹ *American Mathematical Monthly*, 26, 326, 1919.

and radiating it away at a rate of dM_2/dt simultaneously. For a star picking up matter,

$$P = \frac{c_1}{M^5}, \quad (11)$$

and, under the assumptions previously mentioned, for one losing mass by radiation,

$$P = \frac{c_2}{M^2}, \quad (12)$$

where c_1 and c_2 are constants. Let dP_1/dt and dP_2/dt be the rates of change of P due to the picking-up and radiation of mass, respectively. Differentiating (11) and (12) logarithmically gives

$$\frac{dP_1}{P dt} = -\frac{5dM_1}{M dt}, \quad (11')$$

$$\frac{dP_2}{P dt} = -\frac{2dM_2}{M dt}. \quad (12')$$

The total rate of change of the period, dP/dt , equals $dP_1/dt + dP_2/dt$. Hence, by (11') and (12'),

$$\frac{dP}{dt} = -\frac{P}{M} \left(5 \frac{dM_1}{dt} + 2 \frac{dM_2}{dt} \right). \quad (13)$$

The term dM_1/dt is positive, while dM_2/dt is negative. If

$$-\frac{dM_2}{dt} < 2.5 \frac{dM_1}{dt}, \quad (14)$$

dP/dt is negative and P decreases.

Thus, under the assumptions with which we are working, *if a binary is radiating mass at a rate which is less than 2.5 times the rate at which it is picking it up, the period will decrease.*

It is found, similarly, that a will decrease if

$$-\frac{dM_2}{dt} < 3 \frac{dM_1}{dt}. \quad (15)$$

EFFECT ON THE ECCENTRICITY

We have already stated that a secular decrease of mass leaves the eccentricity invariant; we shall now discuss the change due to the

picking-up of matter. We shall assume that the particles of the resisting medium are mutually at rest, and that the center of gravity of the binary is moving through it with a velocity V_0 . Also, let the two components of the binary be equal.

Let ξ, η, ζ represent a set of rectangular axes fixed in the medium. Let the ξ, η -plane be parallel to the plane of the orbit, and let the ξ -axis be parallel to the line joining the two bodies at the time of perihelion. The co-ordinates of the bodies are $\xi_1, \eta_1, \zeta_1; \xi_2, \eta_2, \zeta_2$, and their components of momentum are $m\xi'_1, m\eta'_1, m\zeta'_1; m\xi'_2, m\eta'_2, m\zeta'_2$. Since the matter picked up has no momentum, these components are not altered instantaneously, and the following equations must be satisfied:

$$\left. \begin{aligned} m(\xi'_1)' &= -m'\xi'_1, & m(\eta'_1)' &= -m'\eta'_1, & m(\zeta'_1)' &= -m'\zeta'_1, \\ m(\xi'_2)' &= -m'\xi'_2, & m(\eta'_2)' &= -m'\eta'_2, & m(\zeta'_2)' &= -m'\zeta'_2. \end{aligned} \right\} \quad (16)$$

Since the matter in the medium is at rest with respect to the ξ, η, ζ -axes, the rate at which each of the bodies is sweeping up matter is proportional to its absolute velocity. Hence, we may write

$$\left. \begin{aligned} m' &= mhV_i, \\ V_i^2 &= \xi_i'^2 + \eta_i'^2 + \zeta_i'^2, \quad (i=1 \text{ or } 2). \end{aligned} \right\} \quad (17)$$

The term mh is essentially constant over an interval of time such as one period; its value is equal to the amount of matter which each body picks up in one period divided by the distance which it traverses in the same interval of time. Substitution of the first of (17) in (16) shows that the resistance is proportional to the square of the absolute velocity and that its direction is opposite to that of the absolute motion.

The equations of motion in the absolute co-ordinates are

$$\left. \begin{aligned} \xi''_1 &= -\frac{k^2 m(\xi_1 - \xi_2)}{r^3} - h\xi'_1 V_1, \\ \xi''_2 &= +\frac{k^2 m(\xi_1 - \xi_2)}{r^3} - h\xi'_2 V_2, \end{aligned} \right\} \quad (18)$$

and similar equations in the other co-ordinates. Let

$$x = \xi_1 - \xi_2, \quad y = \eta_1 - \eta_2, \quad z = \zeta_1 - \zeta_2. \quad (19)$$

By means of (19) we find, from (18), that

$$\left. \begin{aligned} x'' &= -\frac{2k^2mx}{r^3} - X, \\ y'' &= -\frac{2k^2my}{r^3} - Y, \\ z'' &= -\frac{2k^2mz}{r^3} - Z, \end{aligned} \right\} \quad (20)$$

where $X = h(\xi'_1 V_1 - \xi'_2 V_2)$ and $Y = h(\eta'_1 V_1 - \eta'_2 V_2)$, etc. Equations (20) are the equations of motion in the relative co-ordinates. The quantities X, Y, Z are the components of the disturbing acceleration, resolved along the co-ordinate axes. According to F. Tisserand,¹ the variation in e due to these components is

$$\left. \begin{aligned} \frac{de}{dt} &= \frac{\epsilon}{na} [(\cos E + \cos v)(X \sin v - Y \cos v) \\ &\quad - \sin v(X \cos v + Y \sin v)], \quad \epsilon = (1 - e^2)^{\frac{1}{2}}, \end{aligned} \right\} \quad (21)$$

where n is the mean motion, E is the eccentric anomaly, and v is the true anomaly. It will be noticed that Z does not enter in this equation.

Let α, β, γ be the components of V_0 referred to the ξ, η, ζ -axes. Then

$$2\alpha = \xi'_1 + \xi'_2, \quad 2\beta = \eta'_1 + \eta'_2, \quad 2\gamma = \zeta'_1 + \zeta'_2. \quad (22)$$

It is found from (22) and (19), after the latter have been differentiated with respect to t , that

$$\left. \begin{aligned} 2\xi'_1 &= 2\alpha + x', & 2\xi'_2 &= 2\alpha - x', \\ 2\eta'_1 &= 2\beta + y', & 2\eta'_2 &= 2\beta - y', \\ \zeta'_1 &= \zeta'_2 = \gamma, & (\text{since } z' &\equiv 0). \end{aligned} \right\} \quad (23)$$

¹ *Traité de mécanique céleste*, 4, 229, 1896.

From the properties of Keplerien motion we know that

$$\left. \begin{aligned} x' &= -\frac{na \sin E}{1-e \cos E}, \\ y' &= \frac{\epsilon na \cos E}{1-e \cos E}, \\ \sin v &= \frac{\epsilon \sin E}{1-e \cos E}, \\ \cos v &= \frac{\cos E - e}{1-e \cos E}, \\ nt &= E - e \sin E. \end{aligned} \right\} \quad (24)$$

By means of (23) and (24) it is possible to eliminate t, v , and the derivatives of the absolute co-ordinates from (21). It is found, after performing the various substitutions, that δe , the change in e in one period, is

$$\delta e = -\frac{\epsilon^2 h(G+H)}{n},$$

where

$$\left. \begin{aligned} G &= \int_0^{2\pi} (V_1 + V_2) \cos E \, dE, \\ H &= \frac{1}{na\epsilon^2} \int_0^{2\pi} (V_1 - V_2) [\beta\epsilon(1 + \cos^2 E - 2e \cos E) - \alpha \sin E \cos E] dE. \end{aligned} \right\} \quad (25)$$

We shall now show that G is always greater than zero, proving first, however, that when regarded as functions of E

$$V_1(E) + V_2(E) \geq V_1(E + \pi) + V_2(E + \pi), \quad \left(+\frac{\pi}{2} \leq E \leq -\frac{\pi}{2} \right). \quad (26)$$

Let U_1 and U_2 be vectors representing the velocity of the two components with respect to the center of gravity; V_1 and V_2 vectors representing their absolute velocities; and V_0 a vector representing the absolute velocity of the center of gravity. Then

$$V_1 = U_1 + V_0, \quad V_2 = U_2 + V_0.$$

Since the components are equal in mass, $U_1 = -U_2$ by symmetry. From the solution of the problem of two bodies we know that

$$|U_1(E)| \geq |U_1(E+\pi)|, \quad \left(+\frac{\pi}{2} \geq E \geq -\frac{\pi}{2}\right).$$

By the symmetry of the ellipse the two vectors in the foregoing equation are parallel. Thus, the vectors $U_1(E)$, $U_2(E)$, $U_1(E+\pi)$, and $U_2(E+\pi)$ are all parallel to each other. Let c and d be the magnitude of the components, respectively, of V_0 when resolved along them and perpendicularly to them. Let u be the magnitude of $U_1(E)$ and w be the magnitude of $U_1(E+\pi)$. Inequality (26) now becomes

$$[(c+u)^2+d^2]^{\frac{1}{2}} + [(c-u)^2+d^2]^{\frac{1}{2}} \geq [(c+w)^2+d^2]^{\frac{1}{2}} + [(c-w)^2+d^2]^{\frac{1}{2}}, \quad (u^2 \geq w^2).$$

By virtue of the relation between u^2 and w^2 this can be reduced to another inequality,

$$(c^2+d^2)^2(u^2-w^2)^2 \geq (c^2-d^2)^2(u^2-w^2)^2,$$

which is always satisfied. Hence, equation (26) is also satisfied. Because of this and because of the known properties of the function $\cos E$ it is easy to show that the integral G is always greater than zero. If H were small compared to G then the eccentricity would decrease continually, even if H were negative. From the second of (25) it follows that H equals zero if both α and β are zero, i.e., if the motion of the center of gravity is perpendicular to the orbital plane.

From the second of (17), (23), and the first two of (24) it is found that

$$\left. \begin{aligned} V_1 &= \frac{(I^2+J)^{\frac{1}{2}}}{2(1-e \cos E)^{\frac{1}{2}}}, \\ V_2 &= \frac{(I^2-J)^{\frac{1}{2}}}{2(1-e \cos E)^{\frac{1}{2}}}, \end{aligned} \right\} \quad (27)$$

where $I^2 = n^2 a^2 + 4V_0^2 + (n^2 a^2 - 4V_0^2)e \cos E$, $J = 4na(\epsilon\beta \cos E - \alpha \sin E)$, $V_0^2 = \alpha^2 + \beta^2 + \gamma^2$. Expansion of V_1 and V_2 in powers of J/I^2 shows that the ratio of (V_1+V_2) to (V_1-V_2) is of the order of na/V_0 (α , β , γ , and V_0 are all considered to be of the same order).

Hence, from (25), the ratio of G to H is of the order of $(na/V_0)^2$. The term na is approximately the relative velocity of the two components of the binary, and is indeed equal to it, as may be seen from the first two of (24), if e is zero. It varies inversely as the cube root of the period P . For a binary whose total mass is 2.5 times that of the sun, $na = 290$ km/sec. if $P = 1$ day; 62 if $P = 100$ days; 30 if $P = 1000$ days; and 9 if $P = 100$ years. The term V_0 will, on the average, probably be less than 15 km/sec.

Thus, if it is true that the periods of binaries are being reduced through the gathering of interstellar matter, then the eccentricities of short-period spectroscopic binaries should be small. In the case of visual binaries whose periods are of the order of 100 years or more we may expect a wide divergence in the values of the eccentricities since the absolute velocity of the center of gravity is comparable to the relative velocity of the components; the effects of close encounters with other stars probably has more influence on the eccentricities of these systems than does the picking-up of matter.

SOME NUMERICAL RESULTS

It may be of interest to compute the time required for a hypothetical visual binary to evolve into a spectroscopic binary. We shall assume that (a) each component has the mass of the sun; (b) the mass radiated by each star is always equal to the mass it picks up, and this is equivalent to the amount of energy radiated by the sun at the present time (assuming that $1 \text{ gm} = 9 \times 10^{20} \text{ ergs}$).

Since two suns are losing and gaining mass,

$$\frac{dM_1}{dt} = -\frac{dM_2}{dt} = 2.6 \times 10^{20} \text{ gm/yr.} \quad (28)$$

Substituting (28) in (13), putting $M = 4 \times 10^{33} \text{ gm}$, and integrating, gives

$$P = P_0 e^{-2 \times 10^{-11} t},$$

where P_0 is the value of the period when $t = 0$, t being expressed in years.

Putting $t = 7 \times 10^{13}$, we find that $P_0/P = 1.2 \times 10^6$. Thus, the period, if it were originally 10,000 years, would be reduced to about 3 days in 70,000,000,000,000 years.

From (12') it is found that the change of the period of a spectroscopic binary due to a secular decrease of mass is of the order of a few millionths of a second per year or less, and from (11') it is seen that the change due to picking up matter would be also undetectable unless this process took place at a rate which is enormous compared to the loss of mass by radiation.

To determine the change in the eccentricity of the binary discussed here we shall make the further assumption that it is at rest relative to the medium. Then $\alpha = \beta = \gamma = V_0 = 0$, and, because of (25) and (27),

$$\delta e = -ah(1 - e^2) \int_0^{2\pi} \cos E \left(\frac{1 + e \cos E}{1 - e \cos E} \right)^{\frac{1}{2}} dE.$$

The integrand of the foregoing expression can be expanded in powers of e . The terms containing even powers of e will carry odd powers of $\cos E$ as factors, and, as these vanish in the integration, it follows that the integral is expansible in odd powers of e . Expanding and integrating, we obtain

$$\delta e = -ah\pi(1 - e^2) \left(e + \frac{3}{8}e^3 + \frac{1}{6}\frac{5}{4}e^5 + \dots \right). \quad (29)$$

If dm_1/dt is the average rate at which each of the components is picking up matter and ds is the element of arc length, then, since $V = ds/dt$, it is found from the first of (17) that

$$\frac{dm_1}{dt} = \int_0^P \frac{dm}{P} = \frac{mh}{P} \int_0^P ds.$$

The last integral on the right-hand side is the expression for the ellipse through which each component moves. If the length of the primeter is expanded as a power series in even powers of e , then

$$\frac{Pdm_1}{dt} = mh\pi a \left(1 - \frac{1}{4}e^2 - \frac{3}{8}\frac{5}{4}e^4 - \dots \right). \quad (30)$$

Eliminating h from (29) by means of (30), it is found, since the average value of de/dt equals $\delta e/P$, that

$$\frac{1}{e} \frac{de}{dt} \left[\frac{(1 - \frac{1}{4}e^2 - \frac{3}{8}\frac{5}{4}e^4 - \dots)}{(1 - e^2)(1 + \frac{3}{8}e^2 + \frac{1}{6}\frac{5}{4}e^4 + \dots)} \right] = -\frac{1}{m} \frac{dm_1}{dt}.$$

Expanding the bracket in even powers of e and integrating gives, neglecting terms of the sixth order and higher,

$$\log \frac{e}{e_0} + \frac{3}{18} (e^2 - e_0^2) - \frac{9}{2 \cdot 5 \cdot 6} (e^4 - e_0^4) = - \frac{t}{m} \frac{dm_1}{dt},$$

e_0 being the value of e when $t=0$. If t is expressed in years, then its coefficient, for the system considered here, is equal to -0.65×10^{-13} . Suppose that $e_0=0.5$. It is found that e/e_0 will equal 0.1, or that e will equal .05 after an interval in t of about 1.5×10^{13} years.

Consider, then, the evolution of a long-period binary which satisfies (14). P , a , and e will decrease continually, and the mass ratio will tend toward unity, since the less massive star swings in a larger orbit and picks up more matter. If in the last stages of its existence it was in a rich region of space, the binary would end up as a spectroscopic system of large mass, high temperature, low density, and early spectral type, such as the O8 system H.D. 1337 (Boss 46). If, on the contrary, it was in a relatively rare region of space, so that its mass remained nearly constant, it would end up with a small mass, low temperature, high density, and late spectral type, such as the F8 system W Ursae Majoris.

The two components of the binary may originally have had rotational motions, and while tidal forces would tend to dampen them, they might, in certain cases, persist. Certain irregularities in the light-curves of some eclipsing binaries may be due to the fact that the components are rotating in planes which do not coincide with the orbital plane.

The data in section iv indicates that tidal forces will disrupt one of the components when the distance between the surfaces is about as large as their diameters ($\sigma=0.5$). Just what will happen to the disrupted star is not easy to say. MacMillan believes that it might form a gaseous ring about the other.¹ The material in this ring may be gradually absorbed by the remaining star, resulting in an early type star, such as O or B, with a high rotational velocity. Another possibility is that the removal of the surface layers of a star may cause a nova.

¹ See, in this connection, an article by O. Struve, *op. cit.*, 73, 94, 1931.

VIII. INTERSTELLAR MATTER

Whether or not equation (14) is satisfied in general by binaries, our present knowledge of interstellar conditions does not permit us to say definitely.

On the assumption that a star loses 1 gm of mass for each 9×10^{20} ergs emitted, we find that the mass of the sun is decreasing at the rate of 4.2×10^{12} gm/sec. MacMillan¹ found that the rate of increase of mass of a single star in passing through a nebulous region is given by

$$\frac{dm}{dt} = \frac{2\pi k^2 m R \rho}{V}, \quad (31)$$

where k^2 = the gravitational constant, m = the star's mass, R = the star's radius, ρ = the density of the medium, and V = the relative velocity between the star and the nebula. For the sun, $R = 7 \times 10^{10}$, $m = 2 \times 10^{33}$, $V = 2 \times 10^6$ (20 km/sec.), all units being in the C.G.S. system. Substituting these values in (31), and putting $dm/dt = 4.2 \times 10^{12}$, we find that ρ must equal 1.4×10^{-19} if the sun is to gather in as much mass as it is radiating away.

The eclipsing binary H.D. 1337 (Boss 46) is the hottest and most massive system whose absolute dimensions are known. From J. A. Pearce's² data we have

$$\begin{aligned} R_1 &= 23.8 \times \odot, \\ R_2 &= 15.5 \times \odot, \\ m_1 + m_2 &= 70 \times \odot, \\ \text{Energy emitted} &= 10^4 \times \odot. \end{aligned}$$

The two components being practically in contact, we shall put $R = 40 \times \odot$. It is found that in order that the total mass should not vary this system must pass through a nebula of density of

$$\rho = 1.7 \times 10^{-23} V.$$

On the assumption that $V = 10$ km/sec.,

$$\rho = 1.7 \times 10^{-17}.$$

¹ *American Mathematical Monthly*, **26**, 326, 1919.

² *Publications of the Dominion Astrophysical Observatory*, **3**, 275, 1927.

When the high luminosity of such stars as H.D. 13,37 is taken into account, it develops that they must be exceedingly rare in space. In general, then, a single star must pass through diffuse matter whose density is of the order of 10^{-19} or 10^{-18} in order that its mass should remain constant.

We have assumed that for the sun $R = 7 \times 10^{10}$ cm, the radius of the photospheric shell. It may well be that a particle of interstellar matter which is intercepted by the corona will fall into the sun. In that case the density required to sustain the sun would be of the order of 10^{-20} gm/cm.

In discussing the densities required to sustain binaries another factor enters. Equation (31) was derived for single stars on the assumption that every particle of matter in the nebula which is attracted by the star, and whose perihelion distance is less than the radius of the star, would fall into it. The semi-major axis of a binary is larger than the sum of the radii of the components, and because of their rapid velocity about each other more matter must be picked up than if the components were pursuing independent paths. Thus, densities of the order of 10^{-20} , or 10^{-21} , or even lower may be sufficient to maintain the masses of binaries.

This does not mean that the density of interstellar matter must be uniformly of about these orders. The existence of luminous and non-luminous nebulae suggests that the density varies widely from point to point. If conditions were right, stars could pick up enough matter in their slow journeys through dense nebulosity to last them through the faster flights through the rarer parts of space. Thus, the life of a star may be of indefinite duration. From the results of the preceding section it follows that if a binary should gain and lose mass alternately, P , a , and e would be reduced.

Interstellar matter may be considered to be of three kinds: (1) atomic, Ca , Na , etc.; (2) fine dust; (3) large bodies, meteors, etc.

J. C. Kapteyn[†] found that the average "effective" mass of a star within the galaxy, i.e., the total mass of a given volume of galactic space divided by the number of luminous stars therein, to be $1.6 \times \odot$. We have already seen that the average mass of one component of a visual binary to be $.88 \times \odot$ (the number of spectroscopic binaries

[†] *Astrophysical Journal*, 55, 302, 1922.

which occur in a given volume of galactic space is very small in comparison with the number of visual binaries). It will be shown in Table IX that most of the stars within 10 parsecs of the sun are dwarfs later in spectral type than the sun, and, therefore, presumably less massive than it. Hence, from this interpretation of Kapteyn's results it appears that the amount of non-luminous matter in the galaxy is of about the same order as that contained within the luminous stars. Eddington¹ obtains a similar result, and deduces 10^{-23} gm/cm to be the average density of interstellar matter.

That interstellar matter of atomic size does exist is well known because of the recent work on the stationary H and K line of *Ca* and the D lines of *Na*. Eddington² and Struve³ find densities of the order of 10^{-24} for the *Ca* and *Na*. Other free atoms may exist, of course, whose presence it would be difficult or impossible to detect. J. S. Plaskett and J. A. Pearce⁴ find that the *Ca* and *Na* atoms partake of the rotation of the galaxy and, since the rotational terms for the stationary lines are one-half that of the stars, deduce that these free atoms cannot form a localized cloud around the star. Nevertheless, this matter may be concentrated into many small clouds which are themselves uniformly distributed.

Further evidence of interstellar matter in the galactic system is obtained from observations of absorption of light. R. J. Trumpler⁵ interprets the absorption as being due to dust particles whose masses are of the order of 2×10^{-19} gm.

With respect to the larger particles, meteors, etc., practically nothing is known since it would take an exceedingly large amount to cause any noticeable absorption.

The large amounts of nebulosity in the galactic system, especially the dark, shows that there exist numerous portions of space where the density of interstellar matter is high enough to produce the massive O- and B-type stars. The close relation between these stars and adjacent or surrounding nebulosity is well known.

¹ *Monthly Notices of the Royal Astronomical Society*, **75**, 375, 1915.

² *Proceedings of the Royal Society*, **A**, **111**, 424, 1926.

³ *Monthly Notices of the Royal Astronomical Society*, **89**, 567, 1929.

⁴ *Ibid.*, **90**, 243, 1930.

⁵ *Astronomical Society of the Pacific*, **42**, 214, 1930.

Summarizing these results, then, we might state:

i) We cannot as yet determine from observation alone whether or not, in general, single stars and binaries are picking up enough interstellar matter to compensate for the loss by radiation, if any.

ii) For the galaxy, the amount of interstellar matter may be comparable in mass to the amount contained within the luminous stars.

iii) There probably exist nebulae of density high enough to form stars of type B and O.

According to the cosmological ideas of MacMillan, the energy of a star is obtained by the annihilation of its mass, and radiant energy in turn is condensed, so to speak, into matter. If the foregoing processes are taking place in our galaxy, and the mass of the galaxy were constant, it would be reasonable to suppose that it would tend to a state of equilibrium wherein the amount of matter contained in the stars and the amount in the form of nebulae and other interstellar matter would be constant; i.e., the amount of matter lost by radiation by the stars would equal the amount swept up.

Suppose, however, that the mass of the galaxy is not constant; that it is radiating more energy into space than it is receiving from other galactic systems. In that case the amount of matter in the stars and in the form of interstellar matter would gradually decrease, but again we may have a kind of equilibrium whereby the amount of matter picked up by the stars would be only slightly less than the amount radiated.

Let us assume that the rate of loss of mass per gram of the galaxy is the same as that of the sun. The galaxy is probably losing mass at a rate much smaller than this, since interstellar matter is not radiating at all, and because the actual number of massive stars of early spectral type O, B, and A forms only a small percentage of all the stars in the galaxy. Thus, of the 77 stars listed in F. Schlesinger's *General Catalogue of Stellar Parallaxes* (1924) that are within 10 parsecs of the sun, and whose spectral types are known, the distribution is as given in Table IX. With the exception of 1 star, Procyon, all of the foregoing 77 lie on the main sequence.

It is known that the rate of emission of energy of a star increases

with the mass. From Eddington's mass-luminosity diagram¹ we find that this relation can be expressed as follows:

$$\frac{dm}{dt} = -\lambda m^\nu. \quad (32)$$

If m does not change very much, λ and ν can be considered constant. It is found that $\nu = 4.8$ for $m = 0.2 \times \odot$, 3.6 for $m = \odot$, and 1.6 for $m = 10 \times \odot$. Integrating (32) gives

$$t = \frac{\nu - 1}{\lambda m^{\nu-1}} (\mu^{\nu-1} - 1), \quad (\nu \neq 1). \quad (33)$$

where $\mu = m_0/m$, and m_0 is the present mass of the sun ($t = 0$).

TABLE IX

	Oe-B8 B	B9-A3 A	A5-F4 F	F5-G2 G	G5-K4 K	K5-Md M	Total
No.	0	4	5	13	31	24	77
Per cent.	0	5.2	6.5	16.9	40.2	31.2	100

For the sun $dm/dt = -1.3 \times 10^{20}$ gm/yr., and $m_0 = 2 \times 10^{33}$ gm. Substituting these values in (32) and (33), (33) becomes

$$t = (\nu - 1) 1.5 \times 10^{13} (\mu^{\nu-1} - 1). \quad (34)$$

We have found in section vii, under the assumption that a binary picks up matter at the same rate at which it is radiating it away, that 7×10^{13} years is sufficient to change it from a visual to a spectroscopic binary. Putting t in (34) equal to this number and $\nu = 3.6$, it is found that the sun, in this interval of time, will lose .32 of its mass. For $\nu = 3$ and 4 the corresponding decreases are .45 and .27. In view of the fact that the entire galaxy is not losing mass at the same rate as the sun, it follows that the mass of the galaxy will not change very much in the time required to transform a long-period binary into a short-period binary.

¹ *The Interior Constitution of the Stars*, Fig. 2, p. 153, 1926.

SUMMARY

The possibility that short-period spectroscopic binaries have evolved into long-period systems has been investigated statistically. It was found that neither contraction, secular decrease of mass, nor close encounters are sufficient to cause the necessary increase in P , a , and e . The possibility that binaries whose periods are less than 25 years have been formed from long-period binaries by the effects of stellar encounters was shown to be small, in view of the large number of systems in the former class with small eccentricities.

The effect of a resisting medium was discussed, and it was shown that P , a , and e would decrease even if the binary is losing as much mass by radiation as it is picking up. While the average interstellar density of the galaxy is perhaps not high enough to maintain equilibrium of mass, there probably exist many regions which are dense enough to form massive stars.

The uniform average mass of binaries, independent of period, as deduced from column 8 of Table VI, and the uniform increase of e with P , as shown in Tables VII and VIII, lead to the conclusion that binary stars, as a rule, have had a common origin; and any apparent separation of binaries into two classes for which P is less than 55 days and P is greater than 55 days is probably due to the observational methods employed.

CONCLUSION

Irrespective of the truth of any theories on the relation between mass and energy and of the density of interstellar matter, one of the following three conditions must apply to any given binary:

A. The system is not losing any mass because of the radiation of energy.

B. It is radiating mass less than 2.5 times as fast as it is gathering it in.

C. It is radiating mass more than 2.5 times as fast as it is gathering it in.

If either condition A or B is satisfied, then P , a , and e are decreasing. If C is satisfied, then, assuming that there are no forces operat-

ing except those already considered, there should be a statistical correlation between the period P and the total mass M of the form

$$M = \frac{D_1}{P^{\frac{1}{3}}},$$

where D_1 is a constant. But we have seen in section v that such is not the case. Therefore, condition A or B must be satisfied. Thus, if there are no forces acting on a binary which tend to increase P , a , and e other than those due to tides, secular decrease of mass, and close encounters, then, in general, P , a , and e of a binary are decreasing.

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